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as a point circle and a pair of lines.

If we treat the ellipse

$$x^2/a^2+y^2/b^2=1$$

in like manner we see, among other things, that the tangents from a focus are perpendicular and that their chord of contact is the corresponding directrix.

Reciprocally, the ellipse supplements the hyperbola in picturing the imaginary part.

TRIANGULAR RESIDUES.

By ORLANDO S. STETSON, Syracuse University.

The summation of arithmetical progressions which commence with unity and whose differences are 1, 2, 3, etc., leads to the theory of *polygonal numbers.*

Employing the difference 1, we obtain the *triangular numbers.* The πth one is $\frac{1}{2}\pi(\pi+1)$. If ρ be any prime number and the series of the first $\rho-2$ triangular numbers be formed, the least positive remainders (modρ) of the first $\frac{1}{2}(\rho-1)$ of these numbers will be all different. Indeed, if two of the remainders were equal for values π_1 and π_2 of π, we should have

$$(\pi_1-\pi_2)(\pi_1+\pi_2+1)\equiv 0 \quad (\text{mod}\rho),$$

which is impossible. The remainder of the middle [$\frac{1}{2}(\rho-1)$th] term $\frac{1}{8}(\rho^2-1)$ occurs once only, while the remainders of the remaining terms repeat themselves in reverse order.

The $\frac{1}{2}(\rho-1)$ different remainders are called *triangular residues* and the remaining numbers of the series 1, 2, ..., $\rho-1$, are called *triangular non-residues.*

Let λ be the least positive remainder of the middle term $\frac{1}{8}(\rho^2-1)$. Then

$$8\lambda+1\equiv 0 \quad (\text{mod}\rho). \tag{1}$$

Theorem I.[*] *If ρ be any prime number excepting* 3, *the sum of the triangular residues* $\equiv\mu(\text{mod}\rho)$, *where* $2\mu\equiv\lambda(\text{mod}\rho)$.

If $f(\pi)$ denote the sum of $1+3+6+\dots\dots\frac{\pi(\pi+1)}{2}$, then $S_t\equiv f\left(\frac{\rho-1}{2}\right)$, where S_t denotes the sum of all the triangular residues. Now,

$$2f(\pi)=2\sum_{h=1}^{\pi}\tfrac{1}{2}h(h+1)=\sum_{h=1}^{\pi}(h^2+h).$$

Summing the series and multiplying both sides of the congruence by 3, we have

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[*] Proved first by M. Stern, *Journal fur Mathematik*, Bd. 69.

$$\pi(\pi+1)(\pi+2) \equiv 6(t_1+t_2+\dots\dots+t_\pi) \pmod{\rho},$$

t_i denoting a triangular residue.

Putting $\pi = \frac{1}{2}(\rho-1)$, and multiplying both sides by 8, we have

$$3(-1) \equiv 48(t_1+t_2+\dots\dots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}.$$

Hence in all cases, excepting $\rho=3$, we have

$$(2) \qquad -1 \equiv 16(t_1+t_2+\dots\dots+t_{\frac{1}{2}(\rho-1)}) \pmod{\rho}.$$

Applying (1), it follows that

$$t_1+t_2+\dots\dots+t_{\frac{1}{2}(\rho-1)} \equiv \mu \pmod{\rho}.$$

In view of (2), we have

Theorem II. *The sum of the triangular residues is a quadratic residue or non-residue according as* ρ *is of the form* $4m+1$ *or* $4m+3$.

Theorem III. *The sum of the triangular non-residues is always a quadratic residue.*

If S_t denotes the sum of the triangular residues and N_t the sum of the triangular non-residues, it follows that

$$(3) \qquad S_t+N_t \equiv 1+2+\dots\dots+\rho-1, \quad S_t \equiv -N_t \pmod{\rho}$$

and the truth of the theorem follows from (2).

A *bi*-triangular non-residue being defined as the double of a triangular non-residue modulo ρ, we have

Theorem IV. *The sum of all the integers between the limits* $\rho/2$ *and* ρ *is congruent modulo* ρ *(a prime number) to the sum of the bi-triangular non residaes of* ρ.

We have

$$\sum_{\frac{1}{2}\rho+\frac{1}{2}}^{\rho-1} h = \frac{\rho(\rho-1)}{2} - \frac{\rho^2-1}{8},$$

and as a result of the definition of a bi-triangular non-residue and of Theorem I, and (3),

$$N_b \equiv -\lambda \pmod{\rho},$$

N_b denoting the sum of the bi-triangular non-residue of ρ. From the two results we obtain

$$\sum_{\frac{1}{2}\rho+\frac{1}{2}}^{\rho-1} h \equiv N_b \pmod{\rho}.$$